SOLUTION EXERCISE SHEET 2

Exercise 1. **(1)** Since T is injective, we know that $a := T(1) \neq 0$. Define $b = a^{-1}T(i)$. Then, for all $z = x + iy \in \mathbb{C}$ we have that

$$T(z) = xa + aby = a(x + by).$$

by the $\mathbb{R}$ linearity of T. Since T is angle preserving, we know that

$$|a|^2 \operatorname{Re}(\bar{b}) = \operatorname{Re}(T(1)\overline{aa^{-1}T(i)}) = \langle T(1), T(i) \rangle = |T(i)||T(1)|\langle 1, i \rangle = 0.$$

Hence, $\operatorname{Re}(\bar{b}) = 0$ which implies the claim.

(2) Note that for $z = x + iy$ we have

$$|T(1)||T(z)|\langle 1, z \rangle = |a|^2|x + yb|x$$

while

$$|z|\langle T(1), T(z) \rangle = |a|^2\sqrt{x^2 + y^2}x$$

which for non-zero x implies

$$\sqrt{x^2 + y^2} = |x + yb| = \sqrt{(x^2 + y^2|b|^2)}$$

and the claim follows by squaring this relation.

(3) Follows immediately by computing the determinant of the standard matrix representation of the function T viewed as a map on $\mathbb{R}^2$.

Exercise 2. We first show that $p(z)$ has no zeros in the open unit disk. For this, we assume, for contradiction, that there exists $z_0 \in \mathbb{C}$ with $|z| < 1$ such that $p(z_0) = 0$. Then

$$\begin{aligned} c_0 = |c_0 - (1 - z_0)p(z_0)| &= \left| \sum_{j=1}^{n} (c_{j-1} - c_j) z_0^j + c_n z_0^{n+1} \right| \\ &\leq \sum_{j=1}^{n} |(c_{j-1} - c_j)| |z_0|^j + c_n |z_0|^{n+1} \\ &< \sum_{j=1}^{n} (c_{j-1} - c_j) + c_n \\ &= c_0 \end{aligned}$$

which yields the contradiction $c_0 < c_0$. To show that there are no $z \in \mathbb{C}$ with $|z| = 1$ such that $p(z) = 0$, we note the following fact about the complex triangle inequality. For $z_1, z_2 \in \mathbb{C}^*$ one has that

$$|z_1 + z_2| = |z_1| + |z_2| \iff z_1 = r_1 e^{i\theta}, z_2 = r_2 e^{i\theta}$$

i.e. if their "angles" are the same.

Assume now that there exists a $z_0 \in \mathbb{C}$ with $|z_0| = 1$ such that $p(z_0) = 0$. Clearly $p(1) \neq 0$. As above, we compute

$$\begin{aligned} c_0 = |c_0 - (1 - z_0)p(z_0)| &= \left| \sum_{j=1}^{n} (c_{j-1} - c_j) z_0^j + c_n z_0^{n+1} \right| \\ &< \sum_{j=1}^{n} |(c_{j-1} - c_j)| z|^j + c_n |z_0|^{n+1} \\ &= \sum_{j=1}^{n} (c_{j-1} - c_j) + c_n \\ &= c_0 \end{aligned}$$

where the strict inequality comes from the fact that for any complex number z, z and z^2 have the same angle if and only if z is a positive real number.

Exercise 3. First, we recall polynomial division: For two polynoimals p and q with $deg(p) > deg(q)$ there exist unique polynomials w and r with $deg(q) > deq(r)$ and $deg(p) = deg(w) + deg(q)$ such that $p = wq + r$.

Now, if p is constant or of degree 1, then we are done. Hence, we can assume $deg(p) = n \geq 2$. Then, by the fundamental theorem of algebra, p has at least one root z_0. Consider now the polynomial $p_0 := (z - z_0)$. By polynomial division. We have that $p = \widetilde{p} p_0 + r$ for some polynomial $\widetilde{p}$ and r. Furthermore, since p_0 is of degree 1 , we know that r must be a constant. However, given that

$$0 = p(z_0) = \widetilde{p}(z_0) p_0(z_0) + r = r$$

we see that necessarily $r = 0$, i.e. $p = \widetilde{p}(z - z_0)$ for some polynomial $\widetilde{p}$ of degree $n - 1$. By iterating this procedure inductively, we arrive at the desired formula.